Towards the modelling of the Purkinje/ myocardium coupled problem: A well-posedness analysis

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Abstract

The Purkinje network is the specialized conduction system in the heart. It ensures the physiological spread of the electrical wave in the ventricles. In this work, in an insulated heart framework, we model the free running Purkinje system, using the monodomain equation. The intra-myocardium part of the Purkinje fiber is coupled to the ventricular tissue using the bidomain equation. The coupling is performed through the extracellular potential. We discretize the problem in time using a semi-implicit scheme. Then, we write a variational formulation of the semi discrete problem in a non standard weighted Sobolev functional spaces. We prove the existence and uniqueness of the solution of the Purkinje/myocardium semi-discretized problem. We discretize in space by the finite element $P_1 - Lagrange$ and conduct some numerical tests showing the anterograde and retrograde propagation of the electrical wave between the tissue and the Purkinje fibers.

Keywords: Purkinje/myocardium, monodomain/bidomain, semi-implicit scheme, discrete problem, weighted Sobolev spaces

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1. Introduction

Heartbeats are generated and controlled by the cardiac conduction system. Each heartbeat is triggered automatically by the natural pacemaker of the heart called the sinus node. The electrical signal generated in the sinus node extends to the atria and causes their contraction. This allows to propel blood from atria into the ventricles. The electrical pulse is then conducted to the atrioventricular node (AV node) in the middle of the heart. Then, the electrical signal propagates through a rapid conduction system (His/Purkinje network) and activates the ventricles, which in turn contract and propel the blood either to the lungs or to the rest of the body. The rapid conduction system is an "electrical" network consisting of cardiac cells that have specific properties for conductivity and excitability. If the activity of this system is interrupted due to cardiac injury or other pathology, the heart rate is disrupted. In this case, blood flow to the brain and other parts of the body may be weakened. As it plays a very important role in electrical activity, it is natural that it also has a role in some pathological cases. It is the case of the Wolff-Parkinson-white syndrome [1], the left and the right bundle branch block [2, 3], the ventricular fibrillation and drug-induced torsades de pointes [4, 5]. When the His/Purkinje System is present, it is generally modeled with the monodomain model [6, 7, 8, 9, 10], which does not take into account the extracellular parts of cardiac cells. The propagation of the electric wave is described by a non-linear reaction diffusion equations coupled to an ordinary differential equation modeling the ionic activity in cardiac cells. For the modeling of the action potential, two approaches exist, the physiological model [11, 12, 13, 14, 15, 16] and the phenomenological model [17, 18, 19, 20, 21]. The literature about the His-Purkinje/myocardium coupling is not abundant. In [22], this coupling is represented at the discrete level for the bidomain equation. A mathematical analysis of this representation could not be performed since the coupling conditions are not given in the continuous level. In [10], authors provide a representation of the coupling conditions at the continuous level, the effect of the Purkinje on the myocardium is represented by

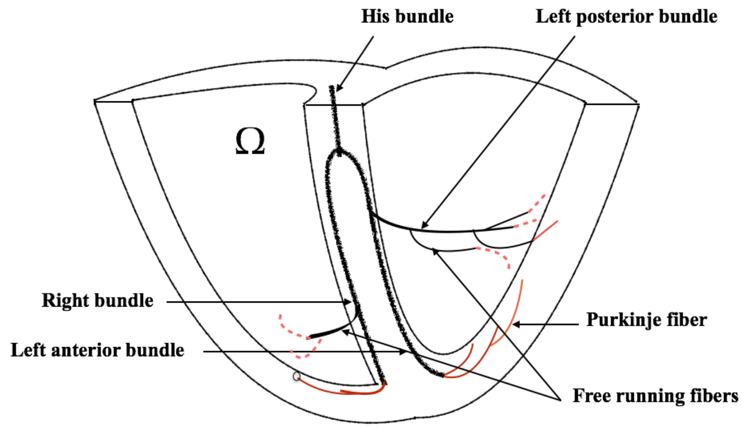

Figure 1: Schematic representation of the specialized conduction system. The red line shows the coupled part. The black line shows the insulated part.

a source term. Whereas, the counter effect is based on a robin-like boundary condition on the terminals of the Purkinje network. The stability of this problem discretized in time and space is the subject of [23].

In [24], the coupling of the Purkinje and the myocardium is performed using the bidomain model for both Purkinje and myocardium. In this paper, we consider the coupling between the Purkinje system and the myocardium with a monodomain/bidomain model and the coupling is carried out by the extracellular potential. We model the extra-myocardium Purkinje fiber, also called free running Purkinje system, using the monodomain equations. The intra-myocardium Purkinje fiber is coupled to the ventricular tissue using the bidomain equations. We write the associated mathematical model under the form of reaction diffusion coupling problem and ordinary differential equations. We discretize the obtained model in time by a semi-implicit Euler scheme. We then prove the existence and uniqueness of the solution at each time step. We make use of weighted Sobolev spaces as in [24, 25]. Then, we discretize in space using the finite element method in a bidimensional framework. We perform some numerical tests in order to assess the anterograde and retrograde propagation of the electrical wave. The outline of the paper is as follow: The model is described in section 2. In section 3, we prove the existence and uniqueness of solution for the time semi-descritized problem. In section 4, we conduct some numerical simulations for the 1D/2D coupled problem.

2. Monodomain/bidomain model for the Purkinje/myocardium coupling in the heart

2.1. Geometry

It is known that the conduction system is made of a set of insulating branches and coupled branches. The insulating branches do not allow an action potential propagation to the surrounding myocardial tissue. They are given by the black lines in the Figure 1. The coupled branches of the Purkinje fibers are given by the red line. We consider a model that describes the propagation of the electrical wave in both Purkinje and myocardium domains. We use a monodomain model in the extra-myocardium Purkinje branches and in the intra-myocardium insulated branches Λ_{isl}. We use a bidomain model in the intra-myocardium coupled Purkinje branches Λ_{cpl}. We suppose that the myocardium occupies a three-dimensional domain Ω and that Purkinje fiber occupies a mono-dimensional part Λ. Without loss of generality, we restrict our study on a segment consisting of both intra- and extra-myocardium fibers as shown in Figure 2.1. We assume that $\Lambda = \Lambda_{\text{isl}} \cup \Lambda_{\text{cpl}} = \{x \in \mathbf{R}^3 \quad x = x(s) \quad , \quad s \in [0, L]\}$, where $x : [0, L] \longrightarrow \mathbf{R}^3$ is a smooth parametrization of Λ. We suppose that the intra-myocardium of Purkinje is $\Lambda_{\text{cpl}} = \{x \in \overline{\Omega}, x = x(s), s \in [s_1, L]\}$ and consequently $\Lambda_{\text{isl}} = x([0, s_1])$, s_1 being a fixed point in $]0, L[$. As in the $1D - 3D$ coupling there is a concern to model the influence of the 1D body, it is appropriate to give some thickness to the fiber operating in Ω. This thickness will play the role of the extracellular environment in our model and makes it more realistic.

We assume that the fiber radius R is a positive constant in time and space. Then we introduce as in [24], the volume occupied by the intra-myocardium

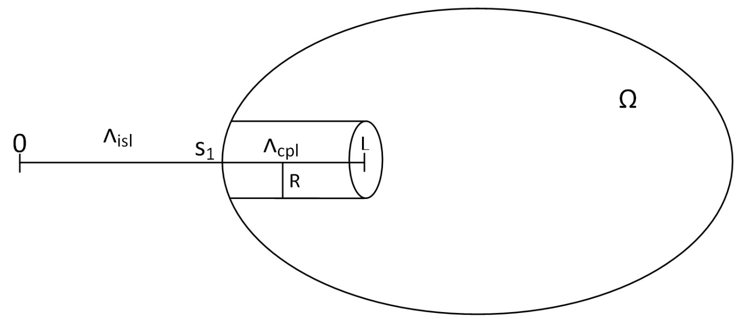

Figure 2: Schematic representation of a Purkinje branch containing both intra and extra myocardium segments.

branch of Purkinje as the set of points in the following cylinder:

$$\Omega^R = \{x \in \mathbf{R}^3 : x = x_0(s,r,\theta), \quad (s,r,\theta) \in (s_1, L) \times [0,R) \times [0,2\pi)\}, \tag{1}$$

We assume that There exists a positive constant R_0 such that for $0 < R < R_0$ we have $\Omega^R \subset \Omega$. In what follows, we consider that $R < R_0$.

Since Λ_{cpl} is compact, the projection from Ω^R to Λ_{cpl} exists and the basic assumption on the fiber geometry is that the projection from Ω^R to Λ_{cpl} is unique:

$$\forall x \in \Omega^R \ : \ \exists! z_0 \in \Lambda_{\text{cpl}} \ : \ dist(x, \Lambda_{\text{cpl}}) = ||x - z_0||. \tag{2}$$

As a consequence we have

$$dist(x_0(s,r,\theta), \Lambda_{\text{cpl}}) = r \qquad \forall (s,r,\theta) \in [s_1, L] \times [0,R) \times [0,2\pi). \tag{3}$$

We will note $d(x) := dist(x, \Lambda_{\text{cpl}})$ for each $x \in \Omega$. The averaging operator on the cross section of a fiber of radius R, perpendicular to the line Λ_{cpl}, is denoted by a bar:

$$\overline{u}(s) := \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R u(s,r,\theta) r dr d\theta. \tag{4}$$

2.2. Mathematical model

In [24], the system of equations in Ω^R is reduced to one dimensional model in two stages. First the three dimensional description of the intracellular current is reduced to one dimension dependent on the curvilinear abscissa. Secondly, an asymptotic expansion for small radius reduces the fiber to one dimension body. Combining the monodomain model used in [10] and the bidomain model introduced in [24], the basic model we advocate writes:

Seek $V_p,\ V,\ \phi_e,\ W,\ W_p$ such that

$$\begin{cases} A_p(C_p\partial_t V_p + I_{ion,p}(V_p,W_p)) = \frac{\partial}{\partial s}(\sigma\frac{\partial V_p}{\partial s}) + \frac{\partial}{\partial s}(\sigma\frac{\partial \overline{\phi_e}}{\partial s})\delta_{\Lambda_{\mathrm{cpl}}} + I_{app,p} \ \ on\ \Lambda\times]0,T[, \\ A(C\partial_t V + I_{ion}(V,W)) = div(\sigma_i\nabla(V+\phi_e)) + I_{app} \ \ in\ \Omega\times]0,T[, \\ -\pi R^2 A_p(C_p\partial_t V_p + I_{ion,p}(V_p,W_p))\delta_{\Lambda_{\mathrm{cpl}}} - A(C\partial_t V + I_{ion}(V,W)) \\ = div(\sigma_e\nabla\phi_e)\ in\ \Omega\times]0,T[, \\ \partial_t W_p + g_p(V_p,W_p) = 0 \ \ on\ \Lambda\times]0,T[, \\ \partial_t W + g(V,W) = 0 \ \ in\ \Omega\times]0,T[, \end{cases} \tag{5}$$

with the following boundary conditions:

$$\begin{cases} \frac{\partial V_p}{\partial s}(0) = 0,\ \frac{\partial V_p}{\partial s}(L) + \frac{\partial \overline{\phi_e}}{\partial s}(L) = 0 \ \ on\ ]0,T[, \\ \frac{\partial \overline{\phi_e}}{\partial s}(s_1) = \big(\frac{\lambda}{1+\lambda}\frac{\partial V_p^{ext}}{\partial s} - \frac{\partial V_p^{int}}{\partial s}\big)(s_1) \qquad on\ ]0,T[, \\ (\sigma_i\nabla V)\cdot n = 0 \qquad\qquad on\ \partial\Omega\times]0,T[, \\ (\sigma_e\nabla\phi_e)\cdot n = 0 \qquad\qquad on\ \partial\Omega\times]0,T[, \end{cases} \tag{6}$$

and initial given conditions

$$V_p(.,0),\ V(.,0),\ \phi_e(.,0),\ W(.,0),\ W_p(.,0). \tag{7}$$

Above A (resp. A_p) defines the surface of membrane per unit volume in the myocardium (resp. Purkinje), C (resp. C_p) represents the capacity per unit of surface in the myocardium (resp. Purkinje), R is the radius fiber for the Purkinje, σ_i is the intracellular conductivity in the myocardium, σ_e is the extracellular conductivity in the myocardium, V is the transmembrane voltage in the myocardium, V_p is the transmembrane voltage in the Purkinje fiber, $\overline{\phi_e}$ is the average extracellular potential in the Purkinje defined by 4, ϕ_e the extracellular potential in the myocardium, I_{ion}(resp. $I_{ion,p}$) is the total membrane current per unit of surface in the myocardium (resp. Purkinje) and W (resp. W_p) is the cell state variables in the myocardium (resp. Purkinje) and $\sigma = \alpha\sigma_i^p$ with $\alpha = 1$

in Λ_{cpl}, $\alpha = \frac{\lambda}{1+\lambda}$ in Λ_{isl}, σ_i^p being the intracellular conductivity of the Purkinje fiber, λ is a positive constant. In this study, the dynamics of W, W_p, $I_{ion,p}$ and I_{ion} are described by the phenomenological two-variables model introduced by Mitchell and Schaeffer [19]. $V_p^{ext} = V_p \mathbf{1}_{\Lambda_{\text{isl}}}$, $V_p^{int} = V_p \mathbf{1}_{\Lambda_{\text{cpl}}}$ and n stands for the outward unit normal on $\partial\Omega$. The definition of I_{ion}, $I_{ion,p}$, g and g_p are given as follows:

$$I_{ion}(V,W) = \frac{W}{\tau_{in}} V^2(V-1) + \frac{V}{\tau_{out}}, \quad I_{ion,p}(V_p, W_p) = \frac{W_p}{\tau_{in}} V_p^2(V_p - 1) + \frac{V_p}{\tau_{out}} \tag{8}$$

$$g(V,W) = \begin{cases} \dfrac{W-1}{\tau_{open}} & \text{if } V \le V_{gate} \\ \dfrac{W}{\tau_{close}} & \text{if } V > V_{gate} \end{cases} \qquad g_p(V_p, W_p) = \begin{cases} \dfrac{W_p - 1}{\tau_{open}} & \text{if } V_p \le V_{gate} \\ \dfrac{W_p}{\tau_{close}} & \text{if } V_p > V_{gate} \end{cases} \tag{9}$$

where the values of the parameters $\tau_{in}, \tau_{out}, \tau_{open}, \tau_{close}, V_{gate}$ are provided in table 1 [19]. We note that σ_i, σ_e are independent of time.

3. Mathematical analysis of the coupled problem

3.1. Functional spaces

In the system 5, a measure term appears in the fourth equation and an averaging operator appears in the second equation which significantly complicate the theoretical and numerical analysis . In particular, the measure term $\delta_{\Lambda_{\text{cpl}}}$ is known not to be in the dual space of $H^1(\Lambda_{\text{cpl}})$, and therefore standard existence and uniqueness results based on this space do not hold for this problem. We will use the functionnal framework proposed in [25]. We suppose that Ω is a connected smooth open domain and for $\alpha \in (-1,1)$, we denote by $L^2_\alpha(\Omega)$ the space of measurable functions u such that

$$\int_\Omega u(x)^2 d^{2\alpha}(x) dx < \infty.$$

where d is the distance defined by 2. This means that $u \in L^2_\alpha(\Omega)$ if and only if $d^\alpha u$ belongs to $L^2(\Omega)$. Equipped with the scalar product

$$(u,v)_{L^2_\alpha(\Omega)} = \int_\Omega u(x) v(x) d^{2\alpha}(x) dx,$$

$L^2_\alpha(\Omega)$ is a Hilbert space. We define the weighted Sobolev space $H^1_\alpha(\Omega)$ by:

$$H^1_\alpha(\Omega) = \{u \in L^2_\alpha(\Omega) : \nabla u \in (L^2_\alpha(\Omega))^3\},$$

provided with it's scalar product

$$(u, v)_{H^1_\alpha(\Omega)} = (u, v)_{L^2_\alpha(\Omega)} + (\nabla u, \nabla v)_{(L^2_\alpha(\Omega))^3}.$$

We recall from [26] that for $\alpha \in (-1, 1)$, the density of smooth functions, Rellich-Kondriatev theorem and Poincaré inequalities hold true in H^1_α.

Remark 1. As Ω is bounded, we have the following injections for $\alpha \in (0, 1)$.

1. $H^1_{-\alpha}(\Omega)$ is continuously embedded in $L^2(\Omega)$, since for $u \in H^1_{-\alpha}(\Omega)$ we have

$$\int_\Omega u^2 dx = \int_\Omega u^2 d^{-2\alpha} d^{2\alpha} dx \leq (\text{diam}(\Omega))^{2\alpha} ||u||^2_{L^2_{-\alpha}(\Omega)}. \tag{10}$$

2. $H^1_{-\alpha}(\Omega)$ is continuously embedded in $H^1_\alpha(\Omega)$, since for $u \in H^1_{-\alpha}(\Omega)$ we have

$$||u||^2_{L^2_\alpha(\Omega)} = \int_\Omega u^2 d^{2\alpha} dx = \int_\Omega u^2 d^{-2\alpha} d^{4\alpha} dx \leq (\text{diam}(\Omega))^{4\alpha} ||u||^2_{L^2_{-\alpha}(\Omega)}. \tag{11}$$

Remark 2. We define an auxiliary distance $\tilde{d}$ from d by :

$$\tilde{d}(x) = min\{d, R\} = \begin{cases} d & in\ \Omega^R, \\ R & elsewhere, \end{cases} \tag{12}$$

$\tilde{d}$ is a Lipschitz function and it is equivalent to the distance d in the sense

$$(\frac{R}{\text{diam}(\Omega)}) d \leq \tilde{d} \leq d \quad on\ \Omega. \tag{13}$$

For any subset $A \subset \Omega$, we define an auxiliary norm by

$$||f||_{\tilde{L}^2_\alpha(A)} := \int_A |f|^2 \tilde{d}^{2\alpha} dx. \tag{14}$$

Of course $||f||_{\tilde{L}^2_\alpha(\Omega^R)} = ||f||_{L^2_\alpha(\Omega^R)}$ and $||.||_{\tilde{L}^2_\alpha(\Omega)}, ||.||_{L^2_\alpha(\Omega)}$ are equivalent:

$$\frac{R^\alpha}{\text{diam}(\Omega)^\alpha} ||f||_{L^2_\alpha(\Omega)} \leq ||f||_{\tilde{L}^2_\alpha(\Omega)} \leq ||f||_{L^2_\alpha(\Omega)}. \tag{15}$$

Finally, the extracellular potential ϕ_e in the myocardium having a null average in Ω, we introduce for $\alpha \in]0,1[$ the subsets

$$V_1 = \{u \in H^1_\alpha(\Omega) : \int_\Omega u(x)dx = 0\} \tag{16}$$

$$V_2 = \{u \in H^1_{-\alpha}(\Omega) : \int_\Omega u(x)dx = 0\}. \tag{17}$$

As it is classical for the weighted Sobolev spaces, we have the following lemma.

Lemma 1. *The spaces V_1, V_2 are two Hilbert spaces when endowed with the norms of $H^1_\alpha(\Omega)$, $H^1_{-\alpha}(\Omega)$ respectively. Moreover we have:*

$$||v||_{H^1_\alpha(\Omega)} \leq C_p||\nabla v||_{L^2_\alpha(\Omega)}, \qquad \forall v \in V_1 \tag{18}$$

and

$$||v||_{H^1_{-\alpha}(\Omega)} \leq C_p||\nabla v||_{L^2_{-\alpha}(\Omega)} \qquad \forall v \in V_2 \tag{19}$$

where c_p is the Poincaré constant which depends only on the connected domain Ω.

PROOF. From 10 we have

$$|\int_\Omega u(x)dx| \leq (mes\ \Omega)^{\frac{1}{2}}||u||_{L^2(\Omega)} \leq (mes\ \Omega)^{\frac{4\alpha+1}{2}}||u||_{L^2_{-\alpha}(\Omega)}.$$

So the function φ defined on $H^1_{-\alpha}(\Omega)$ by $\varphi(u) = \int_\Omega u(x)dx$ is continuous and $V_2 = \varphi^{-1}(\{0\})$ is closed in $H^1_{-\alpha}(\Omega)$. For V_1, we decompose the integral and have in one hand

$$|\int_{\Omega\setminus\Omega^R} u(x)dx| = |\int_{\Omega\setminus\Omega^R} u(x)d^\alpha d^{-\alpha}dx| \leq R^{-\alpha}(mes\ \Omega)^{\frac{1}{2}}||u||_{L^2_\alpha(\Omega)}.$$

In another hand, we have

$$|\int_{\Omega^R} u(x)dx| = |\int_0^{2\pi}\int_{s_1}^L\int_0^R u(x_0(r,s,\theta))r^{\alpha+\frac{1}{2}}r^{-\alpha+\frac{1}{2}}drdsd\theta|$$

$$\leq \int_0^{2\pi}\int_{s_1}^L(\int_0^R u^2r^{2\alpha+1}dr)^{\frac{1}{2}}(\int_0^R r^{-2\alpha+1}dr)^{\frac{1}{2}}dsd\theta \tag{20}$$

$$=\frac{R^{-\alpha+1}}{\sqrt{2-2\alpha}}\int_0^{2\pi}\int_{s_1}^{L}(\int_0^R u^2 r^{2\alpha+1}dr)^{\frac{1}{2}}dsd\theta$$
$$\leq R^{-\alpha+1}\sqrt{\frac{\pi(L-s_1)}{1-\alpha}}||u||_{L^2_\alpha(\Omega^R)}.$$

Which concludes the proof of the first part. The proof of the Poincaré's inequalities is the classical one based on the Rellich theorem [27].

3.2. Weak formulation

Multiplying each one of the equations 5 by a test function, applying the Green formula and using the boundary conditions 6, we obtain the following variational formulation for our problem 5:

Find, for each $t\in]0,T[$, $(V_p,V,\phi_e,W_p,W)\in H^1(\Lambda)\times H^1_\alpha(\Omega)\times V_1\times L^2(\Lambda)\times L^2(\Omega)$, such that

$$\begin{cases}\partial_t W_p + g_p(V_p,W_p)=0 \ \ on\ \Lambda,\\ \partial_t W + g(V,W)=0 \ \ in\ \Omega,\\ A_p\big(C_p\int_\Lambda \partial_t V_p\omega + \int_\Lambda I_{ion,p}(V_p,W_p)\omega\big) - \int_\Lambda I_{app,p}\omega \\ \qquad + \int_\Lambda \sigma\frac{\partial V_p}{\partial s}\frac{\partial\omega}{\partial s} + \int_{\Lambda_{\mathrm{cpl}}}\sigma\frac{\partial\overline{\phi_e}}{\partial s}\frac{\partial\omega}{\partial s} = 0 \quad \forall\omega\in H^1(\Lambda),\\ A\big(C\int_\Omega \partial_t V\psi + \int_\Omega I_{ion}(V,W)\psi\big) - \int_\Omega I_{app}\psi \\ \qquad\qquad + \int_\Omega \sigma_i\nabla V\cdot\nabla\psi + \int_\Omega \sigma_i\nabla\phi_e\cdot\nabla\psi \ = 0 \quad \forall\psi\in H^1_{-\alpha}(\Omega),\\ -k_p\big(C_p\int_{\Lambda_{\mathrm{cpl}}}\partial_t V_p\xi + \int_{\Lambda_{\mathrm{cpl}}} I_{ion,p}(V_p,W_p)\xi\big) + \int_\Omega(\sigma_e+\sigma_i)\nabla\phi_e\cdot\nabla\xi \\ \qquad\qquad + \int_\Omega\sigma_i\nabla V\cdot\nabla\xi - \int_\Omega I_{app}\xi = 0 \qquad\qquad \forall\xi\in V_2,\end{cases} \tag{21}$$

$$V_p,\ V,\ \phi_e,\ W,\ W_p\ \ given\ at\ t=0\ ,\ k_p=\pi R^2 A_p$$

We discretize the system 21 in time using a semi implicit scheme that linearizes our problem. At each time step, we have to solve, the following semi-discrete system: Find $(V_p^{n+1},V^{n+1},\phi_e^{n+1},W_p^{n+1},W^{n+1})\in H^1(\Lambda)\times H^1_\alpha(\Omega)\times V_1\times L^2(\Lambda)\times$

$L^2(\Omega)$, such that

$$\begin{cases} W^{n+1} = W^n - \Delta t g(V^n, W^{n+1}) \;\; in \;\; \Omega, \\ W_p^{n+1} = W_p^n - \Delta t g_p(V_p^n, W_p^{n+1}) \; in \;\; \Lambda, \\ A_p(C_p \int_\Lambda (V_p^{n+1} - V_p^n)\omega + \Delta t \int_\Lambda I_{ion,p}(V_p^n, W_p^{n+1})\omega) - \Delta t \int_\Lambda I_{app,p}^{n+1}\omega \\ \qquad\qquad + \Delta t(\int_\Lambda \sigma \frac{\partial V_p^{n+1}}{\partial s}\frac{\partial \omega}{\partial s} + \int_{\Lambda_{\mathrm{cpl}}} \sigma \frac{\partial \overline{\phi_e^{n+1}}}{\partial s}\frac{\partial \omega}{\partial s}) = 0 \quad \forall \omega \in H^1(\Lambda), \\ A(C \int_\Omega (V^{n+1} - V^n)\psi + \Delta t \int_\Omega I_{ion}(V^n, W^{n+1})\psi) - \Delta t \int_\Omega I_{app}^{n+1}\psi \\ \qquad\qquad + \Delta t \int_\Omega \sigma_i \nabla V^{n+1} \cdot \nabla \psi + \Delta t \int_\Omega \sigma_i \nabla \phi_e^{n+1} \cdot \nabla \psi \;\; = 0 \quad \forall \psi \in H^1_{-\alpha}(\Omega), \\ -k_p(C_p \int_{\Lambda_{\mathrm{cpl}}} (V_p^{n+1} - V_p^n)\xi + \Delta t \int_{\Lambda_{\mathrm{cpl}}} I_{ion,p}(V_p^n, W_p^{n+1})\xi) + \Delta t \int_\Omega \sigma_i \nabla V^{n+1} \cdot \nabla \xi \\ \qquad\qquad + \Delta t \int_\Omega (\sigma_e + \sigma_i)\nabla \phi_e^{n+1} \cdot \nabla \xi - \Delta t \int_\Omega I_{app}^{n+1}\xi = 0 \qquad\quad \forall \xi \in V_2, \end{cases} \tag{22}$$

$$V_p^0, \; V^0, \; \phi_e^0, \; W^0, \; W_p^0 \;\; given.$$

Remark 3. The two first equations of 22 define explicitly W^{n+1} and W_p^{n+1}, so that these two unknowns appear in the second member in the rest of equations.

The problem 22 writes :

find $(V_p^{n+1}, V^{n+1}, \phi_e^{n+1}) \in H^1(\Lambda) \times H^1_\alpha(\Omega) \times V_1$, such that:

$$a((V_p^{n+1}, V^{n+1}, \phi_e^{n+1}), (\omega, \psi, \xi)) = L(\omega, \psi, \xi) \quad\;\; \forall (\omega, \psi, \xi) \in H^1(\Lambda) \times H^1_{-\alpha}(\Omega) \times V_2, \tag{23}$$

where

$$a((V_p^{n+1}, V^{n+1}, \phi_e^{n+1}), (\omega, \psi, \xi)) = C_p(A_P \int_\Lambda V_p^{n+1}\omega - k_p \int_{\Lambda_{\mathrm{cpl}}} V_p^{n+1}\xi) \tag{24}$$

$$+\Delta t \int_\Lambda \sigma \frac{\partial V_p^{n+1}}{\partial s}\frac{\partial \omega}{\partial s} + AC \int_\Omega V^{n+1}\psi + \Delta t \int_\Omega (\sigma_e + \sigma_i)\nabla \phi_e^{n+1} \cdot \nabla \xi + \Delta t \int_\Omega \sigma_i \nabla V^{n+1} \cdot \nabla \xi$$

$$+\Delta t \int_\Omega \sigma_i \nabla V^{n+1} \cdot \nabla \psi + \Delta t \int_\Omega \sigma_i \nabla \phi_e^{n+1} \cdot \nabla \psi + \Delta t \int_{\Lambda_{\mathrm{cpl}}} \sigma \frac{\partial \overline{\phi_e^{n+1}}}{\partial s}\frac{\partial \omega}{\partial s},$$

$$L(\omega, \psi, \xi) = A_p C_p(\int_\Lambda V_p^n \omega - \pi R^2 \int_{\Lambda_{\mathrm{cpl}}} V_p^n \xi) - \Delta t A_p \int_\Lambda I_{ion,p}(V_p^n, W_p^{n+1})\omega \tag{25}$$

$$+\Delta t k_p \int_{\Lambda_{\rm cpl}} I_{ion,p}(V_p^n, W_p^{n+1})\xi + AC\int_\Omega V^n\psi - \Delta t A\int_\Omega I_{ion}(V^n, W^{n+1})\psi$$
$$+\Delta t\int_\Omega I_{app}^{n+1}\xi + \Delta t\int_\Omega I_{app}^{n+1}\psi + \Delta t\int_\Lambda I_{app,p}^{n+1}\omega.$$

In the next section, we show that a solution exists and is unique at each time-step for the time- discrete equations 23.

3.3. Existence and uniqueness of solution

The basic tools for the proof of our existence are the Necàs theorem [28] and the trace theorem proved in [25] recalled below.

Theorem 2. ***(Neĉas)*** *Let* $\mathbf{G_1}$ *and* $\mathbf{G_2}$ *be two Hilbert spaces,* $F \in \mathbf{G_2'}$ *be a bounded linear functional on* $\mathbf{G_2}$ *and* $a(.,.)$ *be a bilinear form on* $\mathbf{G_1}\times\mathbf{G_2}$ *such that*

$$|a(u,v)| \le C_1||u||_{\mathbf{G_1}}||v||_{\mathbf{G_2}} \qquad \forall (u,v)\in \mathbf{G_1}\times\mathbf{G_2}, \tag{26}$$

$$\sup_{u\in\mathbf{G_1}} a(u,v) > 0 \qquad \forall v\in\mathbf{G_2}, v\neq 0, \tag{27}$$

$$\sup_{||v||_{\mathbf{G_2}}\le 1} a(u,v) \ge C_2||u||_{\mathbf{G_1}} \qquad \forall u\in\mathbf{G_1}, \tag{28}$$

where C_1 *and* C_2 *are positive constants. Then there is a unique* $u\in\mathbf{G_1}$ *such that*

$$a(u,v) = F(v) \qquad \forall v\in\mathbf{G_2},$$

which depends linearly and continuously on F:

$$||u||_{\mathbf{G_1}} \le \frac{1}{C_2}||F||_{\mathbf{G_2'}}.$$

Theorem 3. *Let* $0<\alpha<1$. *There exists a unique linear continuous map*

$$\gamma_{\Lambda_{\rm cpl}} : H^1_{-\alpha}(\Omega) \to L^2(\Lambda_{\rm cpl})$$

such that $\gamma_{\Lambda_{\rm cpl}}\phi = \phi|_{\Lambda_{\rm cpl}}$ *for each smooth function* $\phi \in C^\infty(\Omega)$*. In particular, there exists a positive number* $C_{\Lambda_{\rm cpl}} = C_{\Lambda_{\rm cpl}}(\alpha)$ *such that*

$$||\phi||_{L^2(\Lambda_{\rm cpl})} \leq C_{\Lambda_{\rm cpl}}||\phi||_{H^1_{-\alpha}(\Omega)} \qquad \forall \phi \in H^1_{-\alpha}(\Omega)$$

We will apply the theorem 2 in different functional frames. We decompose the bilinear and linear forms defined in 24-25 in four parts and we set

$$A^1(V_p^{n+1},\omega) = A_pC_p\int_\Lambda V_p^{n+1}\omega + \Delta t\int_\Lambda \sigma\nabla V_p^{n+1}\cdot\nabla\omega, \tag{29}$$

$$A^2(V^{n+1},\psi) = AC\int_\Omega V^{n+1}\psi + \Delta t\int_\Omega \sigma_i\nabla V^{n+1}\cdot\nabla\psi, \tag{30}$$

$$A^3(\phi_e^{n+1},\xi) = \Delta t\int_\Omega(\sigma_e+\sigma_i)\nabla\phi_e^{n+1}\cdot\nabla\xi, \tag{31}$$

$$A^{\rm coup}(V_p^{n+1},V^{n+1},\phi_e^{n+1};\omega,\psi,\xi) = \Delta t\int_{\Lambda_{\rm cpl}}\sigma\nabla\overline{\phi_e^{n+1}}\cdot\nabla\omega \tag{32}$$

$$-k_pC_p\int_{\Lambda_{\rm cpl}}V_p^{n+1}\xi + \Delta t\int_\Omega\sigma_i\nabla\phi_e^{n+1}\cdot\nabla\psi + \Delta t\int_\Omega\sigma_i\nabla V^{n+1}\cdot\nabla\xi,$$

$$B^1(\omega) := < B^1,\omega >_{L^2(\Lambda)} = A_pC_p\int_\Lambda V_p^n\omega - \Delta tA_p\int_\Lambda I_{ion,p}(V_p^n,W_p^{n+1})\omega + \Delta t\int_\Lambda I_{app,p}\omega, \tag{33}$$

$$B^2(\psi) := < B^2,\psi >_{L^2(\Omega)} = AC\int_\Omega V^n\psi - \Delta tA\int_\Omega I_{ion}(V^n,W^{n+1})\psi + \Delta t\int_\Omega I_{app}^{n+1}\psi, \tag{34}$$

$$B^3(\xi) := < B^3,\xi >_{L^2(\Omega)} = -k_pC_p\int_{\Lambda_{\rm cpl}}V_p^n\xi + \Delta tk_p\int_{\Lambda_{\rm cpl}}I_{ion,p}(V_p^n,W_p^{n+1})\xi + \Delta t\int_\Omega I_{app}^{n+1}\xi \tag{35}$$

Removing the indices n, noting $\mathbf{u} = (V_p, V, \phi_e), \mathbf{v} = (\omega,\psi,\xi)$ and using the notations 29-35, we have:

$$a(\mathbf{u},\mathbf{v}) = A^1(V_p,\omega) + A^2(V,\psi) + A^3(\phi_e,\xi) + A^{\rm coup}(V_p,V,\phi_e;\omega,\psi,\xi) \tag{36}$$

$$L(\mathbf{v}) = B^1(\omega) + B^2(\psi) + B^3(\xi). \tag{37}$$

It's clear that A^1 is continuous and coercive on $H^1(\Lambda) \times H^1(\Lambda)$. To treat the terms with A^2 and A^3, we recall from [25] the following technical lemma.

Lemma 4. *Let $\alpha^* \in (0,1)$ and $u \in H^1_\alpha(\Omega)$ be given, with $0 < \alpha < \alpha^*$. Consider the θ-Fourier expansions in local coordinates given by*

$$u(s,r,\theta) = \sum_{k \in \mathbf{Z}} A_0^k(r,s) e^{ik\theta} \qquad \textit{in } \Omega^R, \tag{38}$$

Furthermore, we define

$$\Psi(x) = \Psi(x;u) = \begin{cases} \Psi(r,y;u) = \int_r^R t^{2\alpha-1} A^0(t,y)dt & \textit{in } \ \Omega^R, \\ 0 & \textit{elsewhere}, \end{cases} \tag{39}$$

where y can be either the s or the ϕ local variable, depending on the subdomain of Ω^R x belongs to. There are positive constants C_1, C_2, C_3, dependent only on α^*, such that the following estimates hold for each $\alpha \in (0, \alpha^*]$:

$$||u - A^0||_{L^2_{\alpha-1}(\Omega^R)} \leq C_1 ||\nabla u||_{L^2_\alpha(\Omega)}, \tag{40}$$

$$||\Psi||_{L^2_{-\alpha}(\Omega)} \leq C_2 ||u||_{L^2_\alpha(\Omega)}, \tag{41}$$

$$||d^{2\alpha-1} u \nabla d + \nabla \Psi||_{L^2_{-\alpha}(\Omega^R)} \leq C_3 ||\nabla u||_{L^2_\alpha(\Omega^R)}. \tag{42}$$

Following [25], we have the following lemma that will serve to treat the coupling term 32.

Lemma 5. *Let $\alpha \in (-1,1)$; the linear mapping $K : u \to \overline{u}$ from $H^1_\alpha(\Omega)$ to $L^2(\Lambda_{\mathrm{cpl}})$ is bounded and $||K|| \leq \frac{R^{-\alpha}}{\sqrt{\pi}}$.*

Proposition 1. *Let $\sigma_i \in L^\infty(\Omega)$ and assume that there exists a positive constant $\sigma_{i,min}$ such that $\sigma_i \geq \sigma_{i,min}$ in Ω. Let B be a continuous linear operator* *on $H^1_{-\alpha}(\Omega)$. For $\alpha < \min\left(\alpha^*, \frac{\min(m_1, \Delta t m_3)}{2\max(C_2, C_3)}\right)$, there exists a unique $V \in H^1_\alpha(\Omega)$ satisfying*

$$A^2(V, \psi) = B(\psi), \qquad \forall \psi \in H^1_{-\alpha}(\Omega) \tag{43}$$

where A^2 is defined by 30. Moreover there is a positive number C_1 such that

$$||V||_{H^1_\alpha(\Omega)} \leq C_1 ||B||. \tag{44}$$

Proof. The form A^2 is continuous on $H^1_\alpha(\Omega) \times H^1_{-\alpha}(\Omega)$, since we have

$$\begin{aligned}
|A^2(V, \psi)| &\leq |AC \int_\Omega V\psi dx| + |\Delta t \int_\Omega \sigma_i \nabla V \cdot \nabla \psi dx| \\
&\leq m_1 | \int_\Omega V d^\alpha d^{-\alpha} \psi dx| + m_2 | \int_\Omega d^\alpha \nabla V \cdot d^{-\alpha} \nabla \psi dx| \\
&\leq m_1 ||V||_{L^2_\alpha} ||\psi||_{L^2_{-\alpha}} + m_2 ||\nabla V||_{L^2_\alpha(\Omega)} ||\nabla \psi||_{L^2_{-\alpha}(\Omega)} \\
&\leq \max(m_1, m_2) ||V||_{H^1_\alpha(\Omega)} ||\psi||_{H^1_{-\alpha}(\Omega)},
\end{aligned}$$

where $m_1 = AC$, $m_2 = \Delta t ||\sigma_i||_{L^\infty(\Omega)}$. For positivity, let $\psi \in H^1_{-\alpha}(\Omega), \psi \neq 0$. As for $\alpha \geq 0$, $H^1_{-\alpha}(\Omega) \subset H^1_\alpha(\Omega)$, we have

$$\sup_{V \in H^1_\alpha(\Omega)} A^2(V, \psi) \geq A^2(\psi, \psi) \geq \min(m_1, \Delta t m_3)(||\psi||^2_{L^2(\Omega)} + ||\nabla \psi||^2_{L^2(\Omega)}) > 0, \tag{45}$$

where $m_3 = \sigma_{i,min}$ is the lower bound for the conductive term σ_i. Looking now for the $\inf\sup$ condition 28. Let $\alpha^* \in (0,1), \alpha \in (0, \alpha^*]$ and let $V \in H^1_\alpha(\Omega)$. The main idea is to take the test function as following

$$\psi(x) = \tilde{d}(x)^{2\alpha} V(x) + 2\alpha \Psi(x), \tag{46}$$

where $\tilde{d}$ is defined in 12 and $\Psi = \Psi(x, V)$ is the auxiliary function introduced in lemma 4. Thanks to 13, 41 and the Cauchy Schwarz inequality, we have for each $\alpha \in (0, \alpha^*]$:

$$||\psi||_{L^2_{-\alpha}(\Omega)} \leq ||V||_{L^2_\alpha(\Omega)} + 2||\Psi||_{L^2_{-\alpha}(\Omega)} \leq m_4 ||V||_{L^2_\alpha(\Omega)}, \tag{47}$$

where $m_4 = 1 + 2C_2(\alpha^*)$. Moreover, since

$$\nabla\psi = \tilde{d}^{2\alpha}\nabla V + 2\alpha(\nabla\Psi + \tilde{d}^{2\alpha-1}V\nabla\tilde{d}),$$

observing that $\tilde{d} = d$ on Ω^R, $\Psi = 0$, $\nabla\Psi = \nabla\tilde{d} = 0$ on $\Omega\backslash\Omega^R$, and using 42 we obtain

$$||\nabla\psi||_{L^2_{-\alpha}(\Omega)} \leq ||\nabla V||_{L^2_\alpha(\Omega)} + 2||\nabla\Psi + d^{2\alpha-1}V\nabla d||_{L^2_{-\alpha}(\Omega^R)} \leq m_5||\nabla V||_{L^2_\alpha(\Omega)},$$

where $m_5 = 1+2C_3(\alpha^*)$, C_2 and C_3 are the constants in estimates 41, 42, hence

$$||\psi||_{H^1_{-\alpha}(\Omega)} \leq C_4||V||_{H^1_\alpha(\Omega)}, \tag{48}$$

where $C_4 = \sqrt{(m_4^2 + m_5^2)}$.

On another hand, we have

$$A^2(V,\psi) \geq \min(m_1, \Delta t m_3)\left[\int_\Omega V^2\tilde{d}^{2\alpha}dx + \int_\Omega |\nabla V|^2\tilde{d}^{2\alpha}dx\right] + 2\alpha\int_{\Omega^R} V\Psi dx$$
$$+2\alpha\int_{\Omega^R}\nabla V\cdot(\nabla\Psi + \tilde{d}^{2\alpha-1}V\nabla\tilde{d})dx.$$

Using the Cauchy Schwarz inequality, we obtain

$$\begin{aligned} A^2(V,\psi) \quad &\geq \quad \min(m_1, \Delta t m_3)\left[||V||^2_{\tilde{L}^2_\alpha(\Omega)} + ||\nabla V||^2_{\tilde{L}^2_\alpha(\Omega)}\right] \\ &\quad -2\alpha||\nabla V||_{\tilde{L}^2_\alpha(\Omega^R)}||\nabla\Psi + \tilde{d}^{2\alpha-1}V\nabla\tilde{d}||_{\tilde{L}^2_{-\alpha}(\Omega^R)} - 2\alpha||V||_{\tilde{L}^2_\alpha(\Omega^R)}||\Psi||_{\tilde{L}^2_{-\alpha}(\Omega^R)}. \end{aligned} \tag{49}$$

Then, using the the inequalities in Lemma 4, we have

$$\begin{aligned} A^2(V,\psi) \quad &\geq \quad \min(m_1, \Delta t m_3)\left[||V||^2_{\tilde{L}^2_\alpha(\Omega)} + ||\nabla V||^2_{\tilde{L}^2_\alpha(\Omega)}\right] - 2\alpha\big(C_3||\nabla V||^2_{\tilde{L}^2_\alpha(\Omega^R)} + C_2||V||^2_{\tilde{L}^2_\alpha(\Omega^R)}\big). \\ &\geq \quad (\min(m_1, \Delta t m_3) - 2\alpha\, max(C_2, C_3))\left[||V||^2_{\tilde{L}^2_\alpha(\Omega)} + ||\nabla V||^2_{\tilde{L}^2_\alpha(\Omega)}\right] \\ &= \quad (\min(m_1, \Delta t m_3) - 2\alpha\, max(C_2, C_3))\, ||V||^2_{\tilde{H}^1_\alpha(\Omega)}. \end{aligned} \tag{50}$$

Using the equivalence between norms 15, we have

$$A^2(V,\psi) \quad \geq \quad (\min(m_1, \Delta t m_3) - 2\alpha\ \max(C_2, C_3))\, \tfrac{R^{2\alpha}}{\mathrm{diam}(\Omega)^{2\alpha}}||V||^2_{H^1_\alpha(\Omega)}. \tag{51}$$

The third condition (28) in the Nec̀as theorem holds if

$$(\min(m_1, \Delta t m_3) - 2\alpha\ \max(C_2, C_3)) > 0.$$

This is true if

$$\alpha < \min\left(\alpha^*, \frac{\min(m_1, \Delta t m_3)}{2\max(C_2, C_3)}\right).$$

The three conditions of the Neсas theorem are now satisfied, then there exist a unique $V \in H^1_\alpha(\Omega)$ solution of (43).

Let's denote by $M_1 = (\min(m_1, \Delta t m_3) - 2\alpha \max(C_2, C_3)) \frac{R^{2\alpha}}{\text{diam}(\Omega)^{2\alpha}}$, using the linearity of the operator B and the inequality (48), we have

$$||V||_{H^1_\alpha(\Omega)} \leq \frac{C_4}{M_1}||B||. \tag{52}$$

Proposition 2. *Let $\sigma_e \in L^\infty(\Omega)$, $\sigma_i \in L^\infty(\Omega)$ and assume that there exist a positive constants $\sigma_{e,min}$ and $\sigma_{i,min}$ such that $\sigma_e \geq \sigma_{e,min}$ and $\sigma_i \geq \sigma_{i,min}$ in Ω. Let B be a continuous linear operator on V_2. Then, there is a constant $\delta_2 \in (0,1)$ such that for each $\alpha \in (0, \delta_2)$, there exists a unique $\phi_e \in V_1$ satisfying*

$$A^3(\phi_e, \xi) = B(\xi), \qquad \forall \xi \in V_2 \tag{53}$$

where A^3 is defined by 31. Moreover there is a positive number C_2 such that

$$||\phi_e||_{V_1} \leq C_2||B|| \tag{54}$$

Proof. The continuity of A^3 on $V_1 \times V_2$ is trivial since, if we note $\bar{m} = ||\sigma_e + \sigma_i||_{L^\infty(\Omega)}$, we have

$$\begin{aligned} A^3(\phi_e, \xi) = |\Delta t \int_\Omega (\sigma_e + \sigma_i)\nabla\phi_e^{n+1} \cdot \nabla\xi| &\leq T\overline{m}||\nabla\phi_e||_{L^2_\alpha(\Omega)}||\nabla\xi||_{L^2_{-\alpha}(\Omega)} \\ &\leq T\overline{m}||\phi_e||_{V_1}||\xi||_{V_2}. \end{aligned}$$

For positivity, we notice that $V_2 \subset V_1$ and then for $\xi \neq 0$

$$\sup_{\phi_e \in V_1} A^3(\phi_e, \xi) \geq A^3(\xi, \xi) \geq \Delta t(\sigma_{e,min} + \sigma_{i,min})||\nabla\xi||^2_{L^2(\Omega)} > 0.$$

For the inf sup condition, we take again the test function given by 46 but modified so that it has zero mean. We take for a given $\phi_e \in V_1$ as in 46

$$\xi = \tilde{d}^{2\alpha}\phi_e + 2\alpha\Psi,$$

and

$$\tilde{\xi} = \xi - \frac{1}{mes(\Omega)} \int_{\Omega} \xi(x) dx. \tag{55}$$

We have from 48: $\nabla \xi = \nabla \tilde{\xi} \in L^2_{-\alpha}(\Omega)$ and similarly to the proof of lemma 1, $\tilde{\xi} \in L^2_{-\alpha}(\Omega)$. So $\tilde{\xi} \in V_2$ and by the Poincaré inequality 19 we have

$$||\tilde{\xi}||_{L^2_{-\alpha}(\Omega)} \leq C_p ||\nabla \xi||_{L^2_{-\alpha}(\Omega)}. \tag{56}$$

Through 46, 47, 48 and 56 we have, for any $\alpha \in]0, \alpha^*[$, α^* fixed in $]0,1[$:

$$||\tilde{\xi}||_{H^1_{-\alpha}(\Omega)} \leq C||\phi_e||_{H^1_{\alpha}(\Omega)} \tag{57}$$

where C depends only on the Poincaré constant C_p and on α^*. Using now $\tilde{\xi}$ as test function, we get

$$A^3(\phi_e, \tilde{\xi}) = \Delta t \int_{\Omega} (\sigma_e + \sigma_i) \nabla \phi_e \cdot \nabla \tilde{\xi}$$

$$= \Delta t \int_{\Omega} (\sigma_e + \sigma_i) |\nabla \phi_e|^2 \tilde{d}^{2\alpha} dx + 2\alpha \int_{\Omega^R} (\sigma_e + \sigma_i) \nabla \phi_e \cdot (\nabla \Psi + \tilde{d}^{2\alpha - 1} \phi_e \nabla \tilde{d}) dx$$

$$\geq \Delta t \underline{m} ||\nabla \phi_e||^2_{\tilde{L}^2_{\alpha}(\Omega)} - 2\alpha \Delta t \overline{m} C_3 ||\nabla \phi_e||^2_{\tilde{L}^2_{\alpha}(\Omega^R)}$$

$$\geq \Delta t(\underline{m} - 2\alpha \overline{m} C_3) ||\nabla \phi_e||^2_{\tilde{L}^2_{\alpha}(\Omega)}. \tag{58}$$

$$\geq \Delta t(\underline{m} - 2\alpha \overline{m} C_3) \frac{R^{2\alpha}}{2\text{diam}(\Omega)^{2\alpha}} (||\nabla \phi_e||^2_{L^2_{\alpha}(\Omega)} + C_p^{-2} ||\phi_e||^2_{L^2_{\alpha}(\Omega)})$$

$$\geq \Delta t(\underline{m} - 2\alpha \overline{m} C_3) \frac{R^{2\alpha}}{2\text{diam}(\Omega)^{2\alpha}} \min(1, C_p^{-2}) ||\phi_e||^2_{V_1},$$

where $\underline{m} = \sigma_{e,min} + \sigma_{i,min}$ is the lower bound for the conductive term $\sigma_e + \sigma_i$. Defining for α^* fixed in $]0,1[$ the following α-independent quantity

$$\delta_2 = \min(\alpha*, \frac{\underline{m}}{2\overline{m} C_3})$$

we have for $0 < \alpha < \delta_2$

$$A^3(\phi_e, \xi) \geq M_2 ||\phi_e||^2_{H^1_{\alpha}(\Omega)}, \tag{59}$$

where

$$M_2 = \Delta t(\underline{m} - 2\alpha \overline{m} C_3) \frac{R^{2\alpha}}{2\text{diam}(\Omega)^{2\alpha}} \min(1, C_p^{-2}).$$

From 57 and 59 we get the inf sup condition of the Nečas theorem. This completed the proof of the theorem with

$$||\phi_e||_{V_1} \leq \frac{C_4}{M_2}|B||_{V_2}, \tag{60}$$

Now, we come to the main theorem, we introduce the Hilbert spaces

$$\mathbf{E} = H^1(\Lambda) \times H^1_\alpha(\Omega) \times V_1,$$

$$\mathbf{F} = H^1(\Lambda) \times H^1_{-\alpha}(\Omega) \times V_2.$$

Theorem 6. *We assume that for each $t \in [0, T]$, I_{app} and $I_{app,p}$ are respectively in $L^2(\Omega)$ and $L^2(\Lambda)$. We suppose that σ_i and σ_e are in $L^\infty(\Omega)$ and there exist two positive constants $\sigma_{i,min}$ and $\sigma_{e,min}$ such that $\sigma_i \geq \sigma_{i,min}$ and $\sigma_e \geq \sigma_{e,min}$ in Ω. Also assume that the previous data in time are well posed. Then there is $\delta \in (0,1)$, such that if $\alpha \in (0,\delta)$, there exists an unique $\boldsymbol{u} \in \boldsymbol{E}$ such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}) \qquad\qquad \forall \boldsymbol{v} \in \boldsymbol{F}.$$

Where a and L are defined respectively by 24 and 25. Moreover, there is a number C such that:

$$||\boldsymbol{u}||_{\boldsymbol{E}} \leq C||L||_{F'}.$$

Proof. 1. **Continuity.** The bilinear terms $A^k, k = 1, 2, 3$ are respectively continuous on $H^1(\Lambda) \times H^1(\Lambda)$, $H^1_\alpha(\Omega) \times H^1_{-\alpha}(\Omega)$ and $V_1 \times V_2$. To see that the remaining coupling term is continuous on $\mathbf{E} \times \mathbf{F}$, we remark that $\dfrac{\partial\overline{\phi_e}}{\partial s} = \overline{(\dfrac{\partial\phi_e}{\partial s})}$, and make use of lemma 5 to obtain

$$|\Delta t \int_{\Lambda_{\rm cpl}} \sigma \frac{\partial\overline{\phi_e}}{\partial s} \cdot \frac{\partial\omega}{\partial s}| \leq \Delta t||\sigma||_{L^\infty(\Lambda_{\rm cpl})}||\overline{\phi_e}||_{H^1_\alpha(\Omega)}||\omega||_{H^1(\Lambda)}, \tag{61}$$

$$\leq \frac{R^{-\alpha}}{\sqrt{\pi}}\Delta t||\sigma||_{L^\infty(\Lambda)}||\phi_e||_{H^1_\alpha(\Omega)}||\omega||_{H^1(\Lambda)}.$$

We have also using the theorem 3

$$|k_pC_p\int_{\Lambda_{\rm cpl}} V_p\xi| \le k_pC_p||V_p||_{L^2(\Lambda)}||\gamma_{\Lambda_{\rm cpl}}\xi||_{L^2(\Lambda_{\rm cpl})}, \tag{62}$$
$$\le k_pC_p||V_p||_{H^1(\Lambda)}C_{\Lambda_{\rm cpl}}(\alpha)||\xi||_{H^1_{-\alpha}(\Omega)},$$

and

$$|\Delta t\int_{\Omega}\sigma_i\nabla V\cdot\nabla\xi| \le \Delta t||\sigma_i||_{L^\infty(\Omega)}||V||_{H^1_\alpha(\Omega)}||\xi||_{H^1_{-\alpha}(\Omega)}. \tag{63}$$

Finally, we have

$$|\Delta t\int_{\Omega}\sigma_i\nabla\phi_e\cdot\nabla\psi| \le \Delta t||\sigma_i||_{L^\infty(\Omega)}||\phi_e||_{H^1_\alpha(\Omega)}||\psi||_{H^1_{-\alpha}(\Omega)}. \tag{64}$$

So by grouping 61, 63, 62 and 64, we obtain

$$A^{\rm coup}(V_p,V,\phi_e;\omega,\psi,\xi) \le \max\big(\Delta t||\sigma_i||_{L^\infty(\Omega)}, k_pC_pC_{\Lambda_{\rm cpl}}, \tag{65}$$
$$\frac{R^{-\alpha}}{\sqrt{\pi}}\Delta t||\sigma||_{L^\infty(\Lambda_{\rm cpl})}\big)||\mathbf{u}||_E||\mathbf{v}||_F$$

and $A^{\rm coup}$ is continuous on $\mathbf{E}\times\mathbf{F}$.

2. **Non degeneracy.** Let $\mathbf{v} = (\omega,\psi,\xi)\in\mathbf{F}$. Thanks to the proposition 1-2 and the obvious continuity and coercivity of A^1 on $H^1(\Lambda)$, we can choose $\mathbf{u} = (V_p,V,\phi_e)\in\mathbf{E}$ such that V_p, V, ϕ_e are respectively solution of

$$A^1(V_p,v) = (\omega,v)_{H^1(\Lambda)} \qquad \forall v\in H^1(\Lambda) \tag{66}$$

$$A^2(V,v) = (\psi,v)_{H^1_{-\alpha}(\Omega)} \qquad \forall v\in H^1_{-\alpha}(\Omega), \tag{67}$$

$$A^3(\phi_e,v) = (\xi,v)_{H^1_{-\alpha}(\Omega)} \qquad \forall v\in V_2. \tag{68}$$

So, we have

$$A(\mathbf{u},\mathbf{v}) = A^1(V_p,\omega) + A^2(V,\psi) + A^3(\phi_e,\xi) \tag{69}$$
$$= ||\omega||^2_{H^1(\Lambda)} + ||\psi||^2_{H^1_{-\alpha}(\Omega)} + ||\xi||^2_{V_2},$$

and

$$||V_p||_{H^1(\Lambda)} \le \beta_2||\omega||_{H^1(\Lambda)}, \tag{70}$$

$$||V||_{H^1_\alpha(\Omega)} \le C_1||\psi||_{H^1_{-\alpha}(\Omega)}, \tag{71}$$

$$||\phi_e||_{H^1_\alpha(\Omega)} \le C_2||\xi||_{V_2}. \tag{72}$$

Then we have

$$A^{\mathrm{coup}}(V_p, V, \phi_e; \omega, \psi, \xi) \le \big(\frac{R^{-\alpha}}{\sqrt{\pi}}\Delta t||\sigma||_{L^\infty(\Lambda_{\mathrm{cpl}})}C_2 + k_p C_p C_{\Lambda_{\mathrm{cpl}}}\beta_2\big)||\xi||_{H^1_{-\alpha}(\Omega)}||\omega||_{H^1(\Lambda)}$$
$$+\Delta t||\sigma_i||_{L^\infty(\Omega)}\big(C_2 + C_1\big)||\xi||_{H^1_{-\alpha}(\Omega)}||\psi||_{H^1_{-\alpha}(\Omega)}$$

so

$$a(\mathbf{u}, \mathbf{v}) \ge (1-\theta)||\mathbf{v}||^2_{\mathbf{F}},$$

where

$$\theta = \max\big(\frac{R^{-\alpha}}{\sqrt{\pi}}\Delta t||\sigma||_{L^\infty(\Lambda_{\mathrm{cpl}})}C_2 + k_p C_p C_{\Lambda_{\mathrm{cpl}}}\beta_2, \Delta t||\sigma_i||_{L^\infty(\Omega)}\big(C_2 + C_1\big)\big).$$

Knowing that $||\sigma||_{L^\infty(\Lambda_{\mathrm{cpl}})}$ and $||\sigma_i||_{L^\infty(\Omega)}$ are generally small [24] and by choosing Δt and R sufficiently small so that $\theta < 1$, we conclude that the bilinear form a is non-degenerate.

3. **The inf-sup condition.** Now let $\mathbf{u} = (V_p, V, \phi_e) \in \mathbf{E}$ and consider $\mathbf{v} = (V_p, \psi, \tilde{\psi}) \in \mathbf{F}$ where ψ and $\tilde{\psi}$ are defined respectively by 39 and 55. There exist two constant m_4, m_5, both independent of α, such that

$$||\mathbf{v}||_{\mathbf{F}} \le \max(1, C_4)||\mathbf{u}||_{\mathbf{E}}, \qquad A(\mathbf{u}, \mathbf{v}) \ge \min(M_1, M_2, \beta_2)||\mathbf{u}||^2_{\mathbf{E}},$$

so, we have

$$a(\mathbf{u}, \mathbf{v}) \ge (\min(M_1, M_2, \beta_2) - \theta\max(1, C_4))||\mathbf{u}||^2_{\mathbf{E}}.$$

As we have taken the initial data such as the linear form defined by 25 is continuous on $\mathbf{F}$, Nečas theorem applies with $\alpha \in (0, \min(\delta_1, \delta_2))$, $\sigma_i \ge \sigma_{i,min}$, $\sigma_e \ge \sigma_{e,min}$ and

$$C = \frac{\max(1, C_4)}{\min(M_1, M_2, \beta_2) - \theta\max(1, C_4)}.$$

Remark 4. In the theorem 6, we add the hypothesis saying that the previous data in time is well posed in order to ensure the continuity of the linear term L. This continuity is assured if:

1. I_{app} and $I_{app,p}$ are respectively in $L^2(\Omega)$ and $L^2(\Lambda)$ for each $t \in [0, T]$,
2. $I_{ion,p}(V_p^n, W_p^{n+1}) \in L^2(\Lambda)$,
3. $I_{ion}(V^n, W^{n+1}) \in L^2(\Omega)$.

The second condition is assured if $(V_p^0, W_p^0) \in H^1(\Lambda) \times L^2(\Lambda)$ and using Sobolev injection $H^1(\Lambda) \subset C^0(\Lambda)$. The third condition is assured if $(V_p^0, V^0, \phi_e^0) \in H^1(\Lambda) \times H^2_\alpha(\Omega) \times H^2_\alpha(\Omega)$. The proof of this could be performed by induction. Indeed, if we suppose that up to the order n we have $(V_p^n, V^n, \phi_e^n) \in H^1(\Lambda) \times H^2_\alpha(\Omega) \times H^2_\alpha(\Omega)$, then the solution $(V_p^{n+1}, V^{n+1}, \phi_e^{n+1})$ exists and is in $H^1(\Lambda) \times H^2_\alpha(\Omega) \times V_1$. To show that $(V_p^{n+1}, V^{n+1}, \phi_e^{n+1}) \in H^1(\Lambda) \times H^2_\alpha(\Omega) \times H^2_\alpha(\Omega)$, it is enough to come back to equations 5 discretized in time.

4. Numerical approximation

We perform numerical tests in a two-dimensional framework. The myocardium domain is represented by a square. We assume that Ω (resp. Λ) is covered by a regular partition τ (resp. τ_p) of simplexes (resp. *edges*) of maximal diameter h (resp. h_p), with N (resp. N_p) nodes, noted x_1 to x_N (resp. $x_{p,1}$ to x_{p,N_p}). The fiber domain Λ_{cpl} is discretized by extracting edges from the two-dimensional mesh of Ω. It is then a collection $\mathbf{I}_h$ of edges I_k of triangles in τ, $\Lambda_{\text{cpl}} = \bigcup_{I_k \in \mathbf{I}_h} I_k$. Consider the space P_h^1 (resp. $P_{h_p}^1$) of continuous linear finite elements on τ (resp. τ_p) and the usual basis functions $\Phi_1^h, ..., \Phi_N^h$ (resp. $\Phi_1^{h_p}, ..., \Phi_{N_p}^{h_p}$) attached to the nodes $x_1, ..., x_N$ (resp. $x_{p,1}, ..., x_{p,N_p}$). The hat functions associated to the nodes of Λ_{cpl} are assumed to be the restriction of the two-dimensional basis functions onto Λ_{cpl}. Following [24], a function, denoted by $r_{\Lambda_{\text{cpl}}} : k \longrightarrow l$, $k \in \{N_p - m + 1, ..., N_p\}$ and $l \in \{1, ..., N\}$, is defined that maps each one-dimensional node index to the corresponding two-dimensional one:

$$\forall k \in \{N_p - m + 1, ..., N_p\}: \quad x_{r_{\Lambda_{\text{cpl}}}(k)} = x_{m,k} \ \ and \ \ \Phi^{h}_{r_{\Lambda_{\text{cpl}}}(k)} = \Phi^{h_p}_k \quad on \ \ \Lambda_{\text{cpl}}.$$

Using $r_{\Lambda_{\text{cpl}}}$, we define the extension matrix, $R_{\Lambda_{\text{cpl}}} \in \mathbf{R}^{N\times m}$, such that

$$(R_{\Lambda_{\text{cpl}}})_{kl} = \begin{cases} 1 & if \ l = r_{\Lambda_{\text{cpl}}}(k), \\ 0 & otherwise. \end{cases} \tag{73}$$

In other hand, we define another function denoted by $\widehat{r}_{\Lambda_{\text{cpl}}} : k \longrightarrow j,\ k \in \{N_p - m + 1, ..., N_p\}$ and $j \in \{1, ..., N_p\}$, that maps each one-dimensional node in Λ_{cpl} index to the corresponding one-dimensional one in Λ:

$$\forall k \in \{N_p - m + 1, ..., N_p\}: \quad x_{\widehat{r}_{\Lambda_{\text{cpl}}}(k)} = x_{j,k} \ \ and \ \ \Phi^{h_p}_{r_{\Lambda_{\text{cpl}}}(k)} = \Phi^{h_p}_k \quad on \ \ \Lambda_{\text{cpl}}$$

Using $\widehat{r}_{\Lambda_{\text{cpl}}}(k)$ we define the extension matrix, $\widehat{R}_{\Lambda_{\text{cpl}}} \in \mathbf{R}^{N_p\times m}$, such that

$$(\widehat{R}_{\Lambda_{\text{cpl}}})_{kl} = \begin{cases} 1 & if \ l = \widehat{r}_{\Lambda_{\text{cpl}}}(k), \\ 0 & otherwise. \end{cases} \tag{74}$$

After such discretization, the system 23 becomes a linear system of equations that can be expressed in matrix form. Let $V_p^k = (V_{p,h_p}^{1,k}, ..., V_{p,h_p}^{N_p,k})$, $V = (V_h^{1,k}, ..., V_h^{N,k})$ and $\phi_e = (\phi_{e,h}^{1,k}, ..., \phi_{e,h}^{N,k})$ be the coefficients of the approximate solution at the time step k. Then, in matrix form, the finite element problem reads, for each time step, as

$$\begin{aligned} &\begin{bmatrix} \Delta t K_p + C_p A_p M_p & 0 & \Delta t \widehat{R}_{\Lambda_{\text{cpl}}} \overline{K_p} \\ 0 & \Delta t K_i + ACM & \Delta t K_i \\ -k_p C_p R_{\Lambda_{\text{cpl}}} \widehat{M_p} \widehat{R}^T_{\Lambda_{\text{cpl}}} & \Delta t K_i & \Delta t (K_e + K_i) \end{bmatrix} \begin{bmatrix} V_p^{n+1} \\ V^{n+1} \\ \phi_e^{n+1} \end{bmatrix} \\ &= \begin{bmatrix} C_p A_p M_p & 0 & 0 \\ 0 & ACM & 0 \\ -k_p C_p R_{\Lambda_{\text{cpl}}} \widehat{M_p} \widehat{R}^T_{\Lambda_{\text{cpl}}} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_p^n \\ V^n \\ \phi_e^n \end{bmatrix} \\ &+ \begin{bmatrix} \Delta t M_p (I_{app,p}^{n+1} - A_p I_{ion,p}(V_p^n, W_p^{n+1})) \\ \Delta t M (I_{app}^{n+1} - A I_{ion}(V^n, W^{n+1})) \\ \Delta t M I_{app}^{n+1} + \Delta t A_p R_{\Lambda_{\text{cpl}}} \widehat{M_p} I_{ion,p}(V_p^n, W_p^{n+1}) \widehat{R}^T_{\Lambda_{\text{cpl}}} \end{bmatrix} \end{aligned} \tag{75}$$

where the block of the matrix are given by

$$(K_p)_{kl} = \int_\Lambda \sigma \frac{\partial \Phi_k^{h_p}}{\partial s} \frac{\partial \Phi_l^{h_p}}{\partial s} ds \qquad k,l = 0, ..., N_p, \tag{76}$$

$$\overline{(K_p)}_{kl} = \int_{\Lambda_{\mathrm{cpl}}} \sigma \frac{\partial \Phi_k^{h_p}}{\partial s} \overline{\frac{\partial \Phi_l^{h_p}}{\partial s}} ds \qquad k = 0, ..., m; l = 0, ..., N, \tag{77}$$

$$(M_p)_{kl} = \int_\Lambda \Phi_k^{h_p} \Phi_l^{h_p} ds \qquad k,l = 0, ..., N_p, \tag{78}$$

$$\widehat{(M_p)}_{kl} = \int_{\Lambda_{\mathrm{cpl}}} \Phi_k^{h_p} \Phi_l^{h_p} ds \qquad k,l = 0, ..., m, \tag{79}$$

$$(M)_{kl} = \int_\Omega \Phi_k^h \Phi_l^h d\Omega \qquad k,l = 0, ..., N, \tag{80}$$

$$(K_i)_{kl} = \int_\Omega \sigma_i \nabla \Phi_k^h \cdot \nabla \Phi_l^h d\Omega \qquad k,l = 0, ..., N, \tag{81}$$

$$(K_e)_{kl} = \int_\Omega \sigma_e \nabla \Phi_k^h \cdot \nabla \Phi_l^h d\Omega \qquad k,l = 0, ..., N. \tag{82}$$

The block matrices in (75) consist of stiffness and mass matrices calculated on the two-dimensional and one-dimensional meshes, and a matrix $\overline{(K_p)}$ that contains the averaging operator (4). In our tests, Ω is the square (1 cm x 1 cm) and Λ is a 1 cm segment. We use a uniform time and space discretization with $\Delta t = 10^{-1}$ ms and $h = 2 \times 10^{-2}$ cm. We add a segment in the top right of the myocardium, the coupling is performed using the same conditions as for the first segment Λ . We stimulate the first segment at its left free extremity.

In Figure 4, we present the results. After 3ms we see the propagation in the Purkinje fiber (panel (a)). In panel (b), we see how the fiber activates the myocardium in the down left coupling region. The electrical wave propagates through the myocardium (panels (c,d)). When the electrical wave arrives at the top right corner, it activates the second segment of the Purkinje (panel (e)). In Figure 4, we show the plateau phase in panel (a) and the repolarization in panels (b), (c), (d) and (e). In Figure 4, we see the propagation of the extracellular potential in the myocardium at both depolarization and repolarization phase.

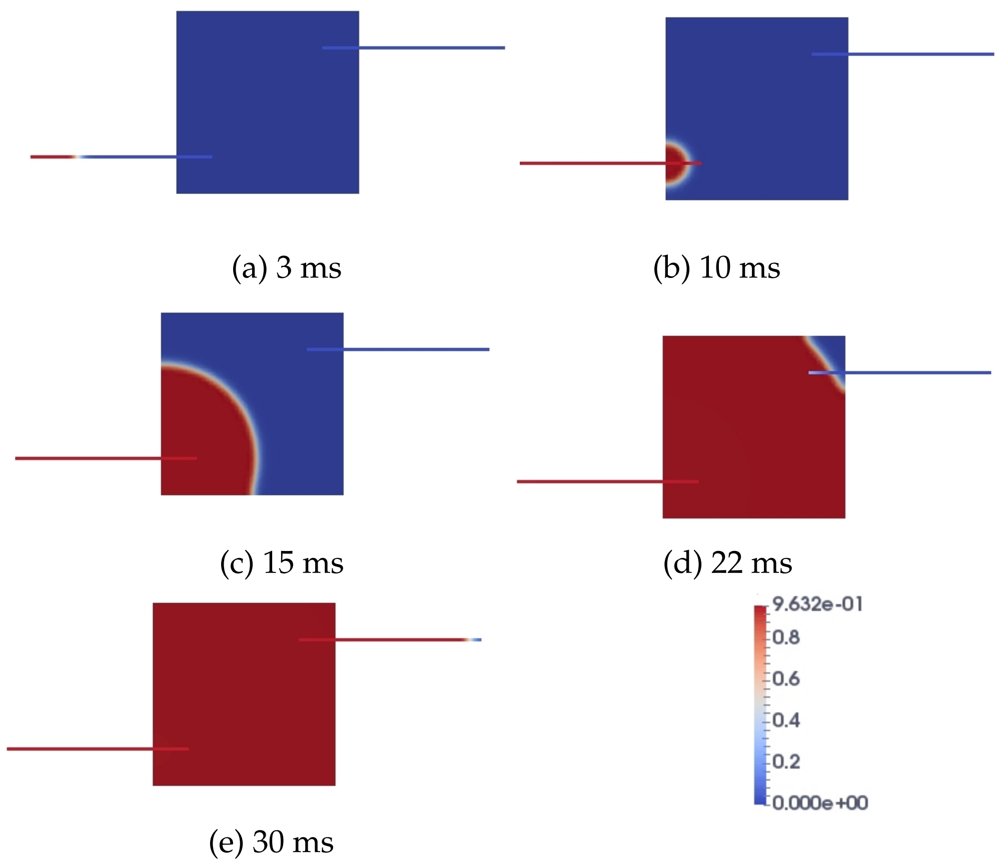

Figure 3: Snapshots of the depolarization phase of the electrical wave showing the anterograde and retrograde circulation of the electrical wave between Purkinje and myocardium.

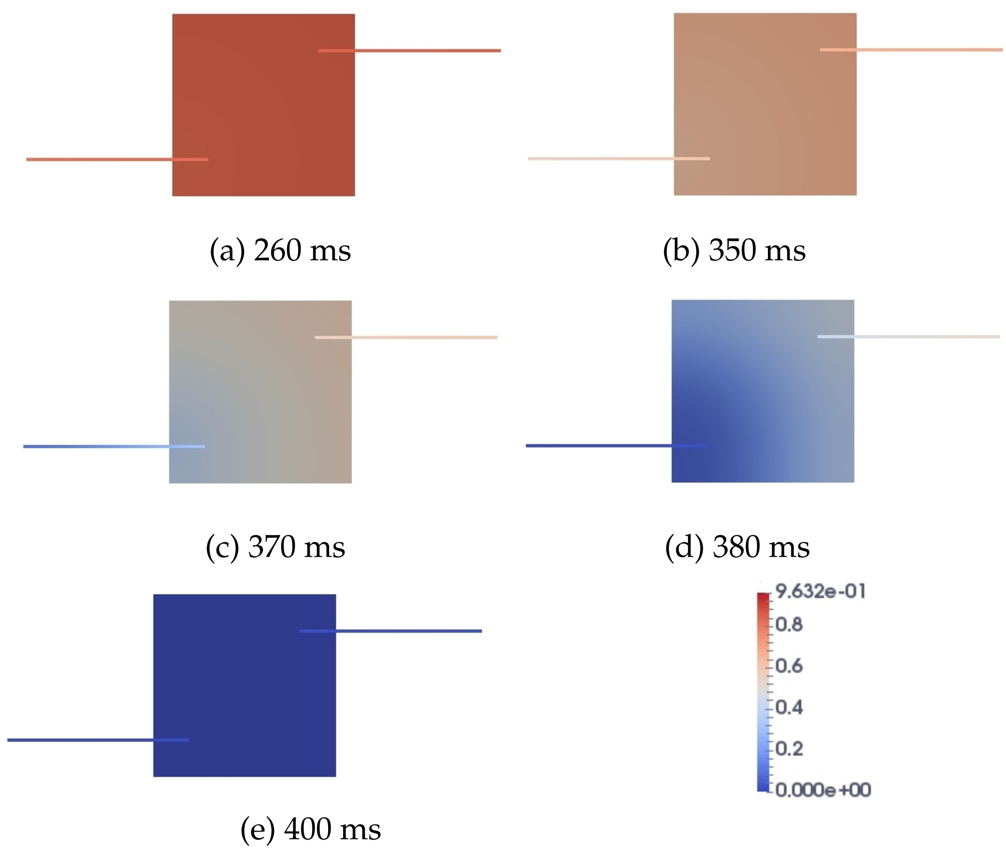

Figure 4: Snapshots of the electrical potential at the plateau phase (panel (a)) and at the repolarization phase (panels (b,c,d,e)).

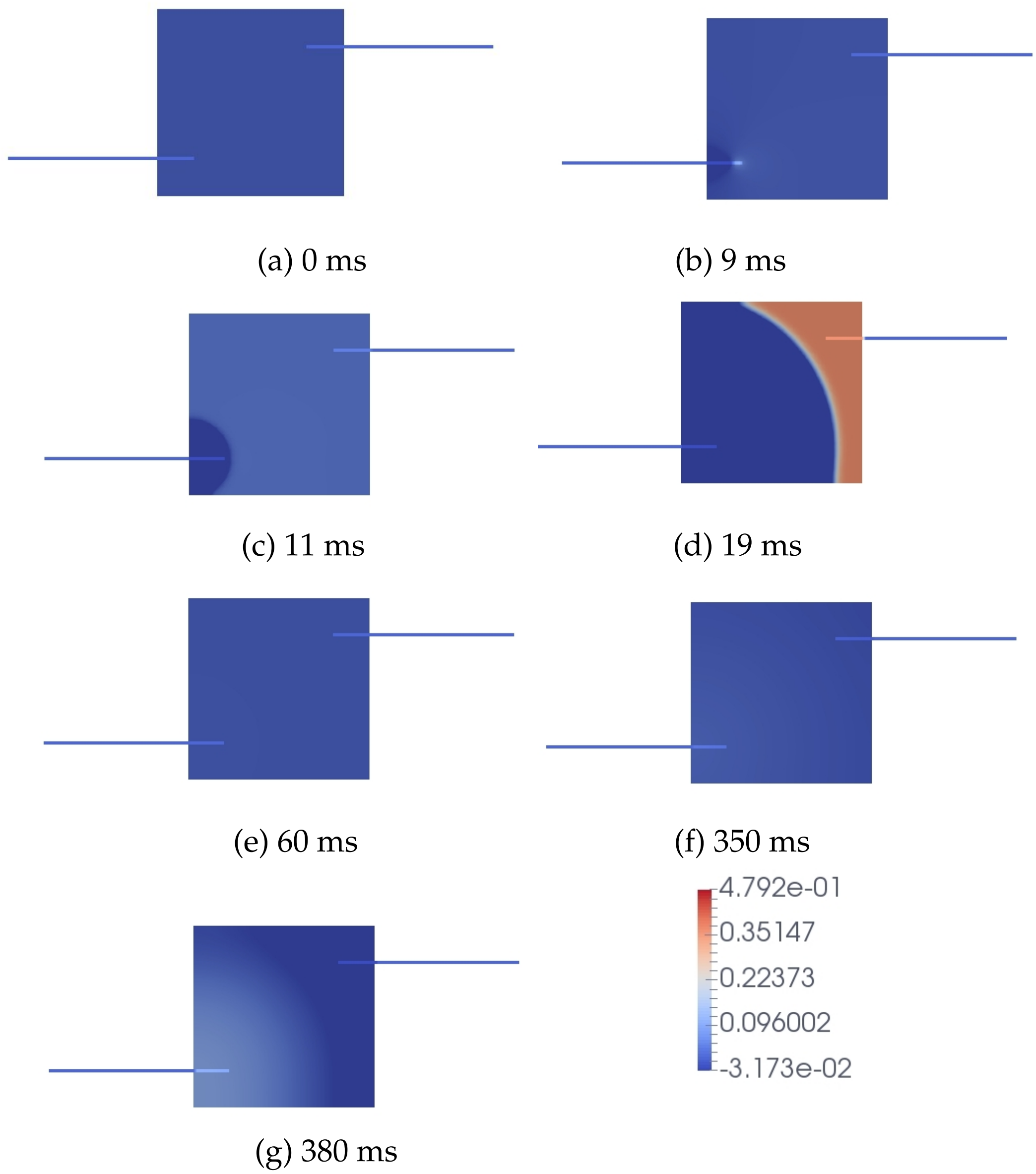

Figure 5: Snapshots of the extracellular potential at the different phase: depolarization and repolarization.

5. Conclusion

In this paper, we have shown the existence and uniqueness of a solution for a monodomain/bidomain coupling problem that models the Purkinje/myocardium conduction system. We realized numerical tests in a two-dimensional framework. The calculation should be extended to the more realistic three-dimensional framework and the physiological ionic model.

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